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Orthogonalization of the SU(n) ⊃ SO(n) projected states and SU(3) ⊃ U(2) isofactors

SigitasErvydas Ališauskas
Institute of Theoretical Physics and Astronomy, Goštauto 12, Vilnius 2600, Lithuania

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Abstract. The explicit direct and inverse orthogonalization coefficients (OCs) are derived for the projected basis states of the two parametric covariant irreducible representations of SU(n) ⊃ SO(n) (including Elliott's $n = 3$ case) and SU(4) ⊃ SU(2)×SU(2) (Draayer), with the alternative choices of the Gram–Schmidt processes, began, respectively, from the lowest (Vergados) or highest absolute value of the intrinsic parameter K. The different direct and inverse OCs (the latter equivalent to the definite boundary SU(n) ⊃ SO(n) isofactors, or the boundary resubducing coefficients) are found to be connected by the definite analytical continuation–contraction procedures with mutually related OCs of the biorthogonal SU(3) × SU(3) ⊃ SU(3) states and the boundary orthonormal pseudocanonical and paracanonical isofactors of SU(3) ⊃ U(2). The orthogonalization coefficients considered are presented as definite algebraic-polynomial structures under the square root, with the linear factors and numerator–denominator polynomials in terms of the partition-dependent functions of Biedenharn and Louck, $A_\lambda\binom{a,b,d,e}{c}$.

1. Introduction

The non-canonical chains of subgroups, such as SU(n) ⊃ SO(n) and, in particular, the SU(3) ⊃ SO(3) and SU(4) ⊃ SU(2) × SU(2) cases, have found many applications in physics and, especially, in nuclear theory. Serious mathematical problems are presented by the appearance of the repeating irreducible representations (irreps) of subgroups for such restrictions, together with the diversity of non-orthogonal analytical versions of the basis states, with the biorthogonal bases included. In a series of previous papers [1, 2], explicit analytical expressions have been proposed for the orthogonalization coefficients of the different versions of the projected (Elliott [3], Draayer [4], see also [5–9]), polynomial [6–9] and their dual (i.e. biorthogonal [7–9]) bases with single-dimensional missing (multiplicity) labels, including the cases of the two-parametric covariant and mixed tensor irreps.

When Louck and co-workers [10, 11] proposed an *ab initio* construction of the ambiguity-free and conjugation-invariant orthogonal basis states for SU(3) ↓ SO(3) (which, however, are not related with the states of any basis, considered in [1, 2]), Ališauskas [1] presented explicit algebraic-polynomial expressions for the direct and inverse orthogonalization coefficients (OCs) of the projected basis states of SU(n) ⊃ SO(n) in the case of the two-parametric irreps. These OCs correspond to the Gram–Schmidt processes, began from the maximal value of the intrinsic projection type parameter K or k, and are expressed in terms of the numerator and denominator polynomials, related with the $A_\lambda\binom{a,b,d,e}{c}$ functions of Biedenharn and Louck [12–14]. Otherwise, these OCs are related by the definite analytical continuation procedure to the OCs of the SU(3) ⊃ U(2) isoscalar factors, which transform the biorthogonal systems [15] of SU(3) ⊃ U(2) isofactors into the orthonormal isofactors of the paracanonical [16] labelling scheme. Note, that the Gram–Schmidt processes began from the maximal values of the multiplicity label were preferred in [1] and [16], since the linearly dependent states of the overcomplete non-orthogonal bases in the both cases appear naturally from below, in accordance with the intermediate expansion technique [6–9, 15].

However, an orthogonal SU(3) $\supset$ SO(3) basis, derived by a Gram–Schmidt process began from the lowest absolute value of the parameter K (for the multiplicity $\leqslant 2$ see Vergados [17], cf Tolstoy [18]) is more convenient and significant from the physicists point of view, since series of states with fixed K and increasing $L \geqslant K$ appear naturally in the nuclear theory. The main purpose of this paper is the explicit analytical orthogonalization of the projected basis states of the two parametric covariant tensor irreps of the chain $\mathrm{SU}(n) \supset \mathrm{SO}(n)$, began from the lowest absolute value of the intrinsic multiplicity label.

Investigations of the polynomial structure [8,19] of the overlaps of the projected states (which were derived by means the intermediate expansion technique [6–8], as well as the overlap structure of their dual basis states [19], lead us to the conclusion that the OCs in this case are related to the OCs of the SU(3) $\supset$ U(2) isofactors, transforming the biorthogonal systems [15] into orthonormal isofactors of the pseudocanonical [15] labelling scheme. In general, the isofactors of this latter scheme are less convenient (less symmetric) than the isofactors of the paracanonical scheme. In these both cases the Gram–Schmidt processes are began from the opposite ends. Otherwise, the pseudocanonical isofactors (after some permutation of their parameters) are equivalent [15] to the U(2)-reduced matrix elements of the SU(3) tensor operators with the null space inclusion property (cf [20]), together with the one-to-one correspondence between their multiplicity labels and the Gelfand–Weyl–Biedenharn pattern [13, 20] of the canonical SU(3) tensor operators. (However, the conclusion of theorem 1.2 of [20] about the isofactors of SU(3) for definite shifts of the SU(2) irrep parameters being equal to 0 is not valid for pseudocanonical isofactors). In their turn, the OCs of the paracanonical and pseudocanonical labelling schemes are also found to be mutually connected by an analytical continuation after substitution of some irrep parameters, equivalent to the hook permutation.

The direct proof of the explicit orthogonalization of the projected $\mathrm{SU}(n) \supset \mathrm{SO}(n)$ basis into the Vergados version (as well as the orthogonalization of the biorthogonal SU(3) $\supset$ U(2) isofactors into the pseudocanonical ones) is very cumbersome, in much the same way as the derivation and proof of the explicit denominator function [13, 20] or OCs presented in [1] and [15]. At first some OCs may be written straightforwardly from the preliminary rearranged expressions of the overlap functions for the initial and final values of the multiplicity label of the orthonormal basis states. In such a way the dependence of the most general OCs on the label of the non-orthogonal states in the linear numerator and denominator factors under the square root sign may also be established. Furthermore, we should examine the polynomial structure and symmetries of the Gram determinants, expanding them in terms of the overlap functions and decomposing in accordance with the null space (ordered vanishing) properties of the OCs. This way leads to expressions for the OCs in terms of the linear numerator and denominator factors under the square root sign and the numerator and denominator polynomials, in which the total degrees in the free parameters are fixed—restricted from above by the initial construction and from below by the distribution of roots of these polynomials (truncated [1, 16] weight space of zeros [20]), also correlated with the null space properties.

Therefore, in order to avoid a lot of trouble, it was expedient to reconsider in section 2 the structure of the boundary paracanonical SU(3) isofactors and their OCs and also to present the explicit OCs for the pseudocanonical labelling scheme, before presenting in section 3 the main result of this paper—the orthogonalization coefficients for the projected basis states of the two-parametric covariant irreps of SU(n) ⊃ SO(n).

In the remainder of the introduction we discuss the notation used for the alternative versions of the projected SU(n) ⊃ SO(n) basis of the two parametric irreps (which cover the general SU(3) ⊃ SO(3) case), taking into account that normalization of [1] and [19] is more convenient for arbitrary $n \geqslant 3$. The definition, some expressions and the main symmetry properties of the partition-dependent functions $A_\lambda\binom{a,b,d,e}{c}$ are also presented.

1.1. Definitions and relations between the projected SU(n) ⊃ SO(n) bases

The two-parametric covariant irreps of SU(n) and SO(n) ($n \geqslant 3$) will be denoted in Introduction and section 3 as $(\lambda\nu\dot{0})$ and $[L_1L_2]$, where $[\lambda+\nu,\nu]$ and $[L_1L_2]$ are the Young frames (partitions). The two versions of the projected non-orthonormal Elliott [3] basis of SU(3) ⊃ SO(3) may be expressed as

$$\left|\begin{matrix}(\lambda\mu)_{E^+}\\ K_+LM\end{matrix}\right\rangle = \mathbf{P}^L_{MK}\left|\begin{matrix}(\lambda\mu)\\ -\frac{1}{3}(2\lambda+\mu)\;\frac{1}{2}\mu\;\frac{1}{2}K_+\end{matrix}\right\rangle \tag{1.1a}$$

$$\left|\begin{matrix}(\lambda\nu)_{E^-}\\ K_-LM\end{matrix}\right\rangle = \mathbf{P}^L_{MK}\left|\begin{matrix}(\lambda\nu)\\ \frac{1}{3}(\lambda+2\nu)\;\frac{1}{2}\nu\;\frac{1}{2}K_-\end{matrix}\right\rangle \tag{1.1b}$$

with the projection operators $\mathbf{P}^L_{MK}$ acting on the intrinsic SU(3) ⊃ U(2) states, denoted as

$$\left|\begin{matrix}(a\ b)\\ \bar{y}_0,\bar{i}_0,i_z=K_+/2\end{matrix}\right\rangle \qquad \text{or} \qquad \left|\begin{matrix}(a\ b)\\ y_0,i_0,i_z=K_-/2\end{matrix}\right\rangle$$

in section 2 (see also [5–7]). The projected (Draayer [4]) states (E) of SU(4) ⊃ SU(2) × SU(2) with the spin S and isospin T (which are expedient to be interchanged in the physical applications) may be obtained by means of the projection operators $\mathbf{P}^S_{M_Sk}$ and $\mathbf{P}^T_{M_Tk}$ acting on the intrinsic SU(4) Gelfand–Tzetlin states:

$$\left|\begin{matrix}(\lambda\nu0)_E\\ kSM_S;kTM_T\end{matrix}\right\rangle = \mathbf{P}^S_{M_Sk}\mathbf{P}^T_{M_Tk}\left|\begin{matrix}\lambda+\nu & & \nu\quad 0\quad 0\\ & k+\frac{1}{2}\lambda+\nu\quad 0\quad 0 & \\ & k+\frac{1}{2}\lambda+\nu\quad 0 & \\ & k+\frac{1}{2}\lambda+\nu & \end{matrix}\right\rangle \tag{1.2}$$

(see also [8] and [21–23]). Irreducible representations $[L_1L_2]$ of SO(4) with $L_1=S+T$ and $L_2=|S-T|$ appear for the basis states of the chain SU(4) ⊃ SO(4), which may be constructed more easy by means of the extrapolated to SU(n) ⊃ SO(n) isofactors [7, 8] for coupling $(p_100)\otimes(p_200)$ to $(\lambda\nu0)$, although a supplementary factor is necessary for renormalization between the extrapolated E basis [8] and Elliott basis E^- in the $n=3$ case (taking into account the correspondence $k=\frac{1}{2}K_-$, $L_1=L$, $L_2=0$ or 1, so that $\lambda-L_1-L_2$ is an even integer).

In [1] the relation between the above-mentioned SU(n) ⊃ SO(n) analogue [8]† E of SU(4) ⊃ SU(2) × SU(2) projected basis E (obtained using the definite SU(n) ⊃ SO(n) isofactors as analytical continuation of the SU(4) ⊃ SO(4) isofactors) and basis $[E]$ used

† For some corrections see also [9]. Note that in the r.h.s. of (5.2) the factor 2^{-n+3} is omitted and $\nu-L_2-\Delta_0$ is even.

in [19] with a more natural behaviour (with respect to the analytical continuation for arbitrary n) was discussed. The new basis $[E]$ was determined as

$$\left| \begin{matrix} (\lambda\nu\dot{0})_{[E]} \\ k[L_1L_2]_n \\ \cdots \end{matrix} \right\rangle \equiv \sum_{\bar\kappa\leqslant k} \begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0}) \\ [p_1] & [p_2] & \bar\kappa[L_1L_2]_n \end{bmatrix} \left| \begin{matrix} (\lambda\nu\dot{0}) \\ \bar\kappa[L_1L_2]_n \\ \cdots \end{matrix} \right\rangle \tag{1.3a}$$

$$= N(\lambda\nu; k[L_1L_2]_n)\, G^{-1}(n)\, [2(2\nu+n-4)!!(2\lambda+2\nu+n-2)!!$$

$$\times (L_1+L_2+n-3)(L_1-L_2+1)]^{-1/2} \left| \begin{matrix} (\lambda\nu\dot{0})_E \\ k[L_1L_2]_n \\ \cdots \end{matrix} \right\rangle \tag{1.3b}$$

where $k = \frac{1}{2}(p_1-p_2)$, $\lambda+2\nu = p_1+p_2$, with the boundary isofactor of $\mathrm{SU}(n) \supset \mathrm{SO}(n)$ on the r.h.s. of (1.3a) appearing as the weight coefficient for expansion in terms of the orthonormal states with the label $\bar\kappa$. (In equation (1.3a) of [1] the corresponding label $\kappa \leqslant k$]. Here and in section 3

$$N(\lambda\nu; k[L_1L_2]_n) = \left[\frac{(\lambda+1)\,(\lambda+2\nu-L_1-L_2)!!(\lambda+2\nu+L_1-L_2+n-2)!!}{(\lambda+2\nu+2k+n-4)!!} \right.$$

$$\left. \times \frac{(\lambda+2\nu-L_1+L_2+n-4)!!(\lambda+2\nu+L_1+L_2+2n-6)!!}{(\lambda+2\nu-2k+n-4)!!} \right]^{1/2}. \tag{1.4}$$

For n even $G(n) = 1$ and for n odd $G(n) = \Gamma(\frac{1}{2}) \big/ \sqrt{2}$. In the $\mathrm{SU}(3) \supset \mathrm{SO}(3)$ and $\mathrm{SU}(4) \supset \mathrm{SU}(2)$

$$\left| \begin{matrix} (\lambda\nu)_{[E]} \\ kL(L_2)M \end{matrix} \right\rangle = \begin{bmatrix} p_1/2 & p_2/2 & \lambda/2 \\ p_1/2 & -p_2/2 & k \end{bmatrix} \begin{bmatrix} p_1 & p_2 & L \\ p_1 & -p_2 & 2k \end{bmatrix}^{-1} \left| \begin{matrix} (\lambda\nu)_{E^-} \\ 2k\,LM \end{matrix} \right\rangle \tag{1.5}$$

$$\left| \begin{matrix} (\lambda\nu 0)_{[E]} \\ k; STM_SM_T \end{matrix} \right\rangle = \begin{bmatrix} p_1/2 & p_2/2 & S \\ p_1/2 & -p_2/2 & k \end{bmatrix}^{-1} \begin{bmatrix} p_1/2 & p_2/2 & T \\ p_1/2 & -p_2/2 & k \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} p_1/2 & p_2/2 & \lambda/2 \\ p_1/2 & -p_2/2 & k \end{bmatrix} \left| \begin{matrix} (\lambda\nu 0)_E \\ kSM_S; kTM_T \end{matrix} \right\rangle \tag{1.6}$$

with the renormalization coefficients expressed in terms of the boundary SU(2) Clebsch–Gordan coefficients of the type

$$\begin{bmatrix} j_1 & j_2 & j \\ j_1 & -j_2 & j_1-j_2 \end{bmatrix} = \left[\frac{(2j_1)!(2j_2)!(2j+1)}{(j_1+j_2-j)!(j_1+j_2+j+1)!} \right]^{1/2}.$$

The boundary isofactors on the r.h.s. of identity (1.3a) for $n = 3$ are multiples of the boundary resubducing coefficients between the chains $\mathrm{SU}(3) \supset \mathrm{U}(2) \supset \mathrm{SO}(2)$ and $\mathrm{SU}(3) \supset \mathrm{SO}(3) \supset \mathrm{SO}(2)$

$$\begin{bmatrix} (p_10) & (p_20) & (\lambda\nu) \\ [p_1] & [p_2] & \bar\kappa L(L_2)_3 \end{bmatrix} = \left[\frac{(\lambda+1)(p_1+p_2-L)!(p_1+p_2+L+1)!}{2^{p_1+p_2}\nu!(\lambda+\nu+1)!(2p_1-1)!!(2p_2-1)!!(2L+1)} \right]^{1/2}$$

$$\times \left\langle \begin{matrix} (\lambda\nu) \\ \frac{1}{3}(\lambda+2\nu)\,\frac{1}{2}\nu\,\frac{1}{2}K_- \end{matrix} \right| \left. \begin{matrix} (\lambda\nu) \\ \bar\kappa\, LK_- \end{matrix} \right\rangle \tag{1.7a}$$

(with $K_- = p_1 - p_2,\ \lambda + 2\nu = p_1 + p_2,\ \bar{\kappa} \leqslant \frac{1}{2}K_-$), and coincide with the orthogonalization coefficients $\widetilde{\mathcal{O}}^{(\lambda\nu,L)_3}_{k\bar{\kappa}}$ of the dual basis $[\overline{E}]$, as well as

$$\left[\begin{array}{ccc} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0}) \\ [p_1] & [p_2] & \bar{\kappa}[L_1L_2]_n \end{array}\right] \equiv \widetilde{\mathcal{O}}^{(\lambda\nu,L_1L_2)_n}_{k\bar{\kappa}} \tag{1.7b}$$

for arbitrary n. Otherwise, the boundary resubducing coefficients [between the chains SU(3) ⊃ U(2) ⊃ SO(2) and SU(3)⊃SO(3) ⊃ SO(2)] on the r.h.s. of (1.7a) are equivalent to OCs of the dual projected basis $\overline{E}$. Hence, the orthonormal SU(n) ⊃ SO(n) states may be expanded as follows:

$$\left|\begin{array}{c} (\lambda\nu\dot{0}) \\ \bar{\kappa}[L_1L_2]_n \\ \cdots \end{array}\right\rangle = \sum_{k\leqslant\bar{\kappa}} \mathcal{O}^{(\lambda\nu,L_1L_2)_n}_{\bar{\kappa}k} \left|\begin{array}{c} (\lambda\nu\dot{0})_{[E]} \\ k[L_1L_2]_n \\ \cdots \end{array}\right\rangle \tag{1.8a}$$

$$= \sum_{k\geqslant\bar{\kappa}} \widetilde{\mathcal{O}}^{(\lambda\nu,L_1L_2)_n}_{k\bar{\kappa}} \left|\begin{array}{c} (\lambda\nu\dot{0})_{[\overline{E}]} \\ k[L_1L_2]_n \\ \cdots \end{array}\right\rangle \tag{1.8b}$$

with the orthogonalization coefficients satisfying the conditions

$$\sum_{\bar{\kappa}\geqslant k\geqslant\bar{\kappa}} \mathcal{O}_{\bar{\kappa},k}\widetilde{\mathcal{O}}_{k,\bar{\kappa}'} = \delta_{\bar{\kappa},\bar{\kappa}'} \qquad \sum_{k\geqslant\bar{\kappa}\geqslant k} \widetilde{\mathcal{O}}_{k,\bar{\kappa}}\mathcal{O}_{\bar{\kappa},k'} = \delta_{k,k'} \tag{1.9}$$

since $\mathcal{O}_{\bar{\kappa},k} = 0$ for $\bar{\kappa} < k$ and $\widetilde{\mathcal{O}}_{k,\bar{\kappa}} = 0$ for $k < \bar{\kappa}$.

The direct and inverse orthogonalization coefficients used both together are very convenient for the Engeland [24] (see also [4] and [9]†) construction of the SU(3) ⊃ SO(3) isofactors in terms of special SU(3) ⊃ U(2) isofactors, the inverse resubducing coefficients [8] and SU(2) Clebsch–Gordan coefficients, as demonstrated by equation (1.12) of [1]. Recall [1] that the orthogonalization coefficients for the E_+ basis may be obtained from OCs of E_- basis of SU(3) ⊃ SO(3), after the substitution of parameters $\lambda \to \mu$ and $\nu \to \lambda$ and appearing additional phase factor $(-1)^{k-\bar{\kappa}}$. The transition to the two-parametric contravariant irreps $(\dot{0}\nu\lambda)$ of SU(n) is accompanied by the same phase factor.

1.2. *Definition and some properties of the Biedenharn and Louck A_λ functions*

In our formulae throughout the paper we will use the following notation for the rising and falling factorials:

$$(X)_k = X(X+1)\cdots(X+k-1) = (X+k-1)^{(k)} \tag{1.10a}$$

$$X^{(k)} = X(X-1)\cdots(X-k+1). \tag{1.10b}$$

with $(X)_0 = X^{(0)} = 1$ and $(X)_1 = X^{(1)} = X$.

In [1, 16] and in this paper the numerator and denominator polynomials of OCs are expressed in terms of the partition-dependent generalization of the Saalschutzian (balanced)

† Note that $\frac{1}{4}$ should be changed (twice) to $\frac{1}{2}$ in (5.5) of [9], the first symbol × on the r.h.s. of (5.10) should be corrected to + and the parameter $|K'| \geqslant \Delta + \delta$.

${}_4F_3$ hypergeometric series [25, 26], introduced by Biedenharn and Louck [12–14] as the auxiliary A_λ function (used in [13] in the proof of the symmetry of the $\mathcal{G}_q^t(A)$ polynomials):

$$A_\alpha\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} = \prod_{s=1}^{t}(a+b+c-s+1)_{\alpha_s}(c+d+e-s+1)_{\alpha_s}$$
$$\times\sum_{\beta\gamma} g(\beta\gamma\alpha)\,\langle {}_2\mathcal{F}_1(a,b;a+b+c)|\beta\rangle\,\langle {}_2\mathcal{F}_1(c+d,c+e;c+d+e)|\gamma\rangle \qquad (1.11a)$$
$$= \sum_{\beta\gamma}\frac{g(\beta\gamma\alpha)}{M(\beta)M(\gamma)}\prod_{s=1}^{t}(a-s+1)_{\beta_s}(b-s+1)_{\beta_s}$$
$$\times(a+b+c+\beta_s-s+1)_{\alpha_s-\beta_s}(c+d+e+\gamma_s-s+1)_{\alpha_s-\gamma_s}$$
$$\times(c+d-s+1)_{\gamma_s}(c+e-s+1)_{\gamma_s}. \qquad (1.11b)$$

Here the Littlewood–Richardson number $g(\beta\gamma\alpha)$ is equal to the multiplicity of the irrep α of $\mathrm{U}(t)$ in the decomposition of the direct product $\beta\otimes\gamma$, the generalized hypergeometric coefficients $\langle {}_2\mathcal{F}_1(a,b;c)|\lambda\rangle$ are expressed as follows:

$$\langle {}_2\mathcal{F}_1(a,b;c)|\lambda\rangle = M^{-1}(\lambda)\prod_{s=1}^{t}(a-s+1)_{\lambda_s}(b-s+1)_{\lambda_s}\Big/(c-s+1)_{\lambda_s} \qquad (1.12)$$

and measure

$$M(\lambda) = N!/d_\lambda = \prod_{s=1}^{t}(\lambda_s+t-s)!\Bigg/\prod_{s<k}(\lambda_s-\lambda_k-s+k) \qquad (1.13)$$

is expressed in terms of the dimensions $N!$ and d_λ of the permutation group S_N and its irrep λ.

The function $A_\alpha\left(\begin{smallmatrix} a,b,d,e \\ c\end{smallmatrix}\right)$ is a polynomial (with rational coefficients for a fixed partition α) in five free parameters a,b,c,d,e and satisfies the evident symmetry relations [12, 14]

$$A_\alpha\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} = (-1)^{\sum_s\alpha_s}A_{\alpha^*}\begin{pmatrix} -a,-b,-d,-e \\ -c \end{pmatrix} \qquad (1.14a)$$
$$= A_\alpha\begin{pmatrix} d+c,e+c,a+c,b+c \\ -c \end{pmatrix} \qquad (1.14b)$$
$$= A_\alpha\begin{pmatrix} b,a,d,e \\ c \end{pmatrix} = A_\alpha\begin{pmatrix} a,b,e,d \\ c \end{pmatrix} \qquad (1.14c)$$

(where α^* means the partition with the interchanged rows and columns of the Young tableau α) and is S_4 invariant [14] with respect to all 4! permutations of the parameters a,b,d,e. (This last invariance is very important for the uniqueness of our solutions for the numerator polynomials [1, 2, 16], although its proof [14]† for the permutation $b\leftrightarrow d$ is very tedious). We get the balanced ${}_4F_3$ hypergeometric series in the case of a single row or (taking into account relation (1.14a)) column in the partition α.

The partitions α of the functions A_α, corresponding to our denominator polynomials, include t equal rows: $\dot{\alpha} = [\widetilde{h^t}]$ (the coefficients of the corresponding polynomials turn into integers after multiplication by $M(\alpha)$), but in the case of the numerator polynomials

† Note that lemma 6.2 of [14] should be reformulated (since z_n and z_n'' are not necessarily equal to 0 both together) without spoiling the main proposition.

only partitions with the deleted squares in the last row, $\alpha = [\widetilde{h}^{t-1}, h]$, or column, $\alpha' = [(\widetilde{h}+1)^{t'-l}\widetilde{h}^{l}]$, may also appear. Hence, in these cases the multiplicities $g(\beta\gamma\alpha)$ or $g(\beta\gamma\alpha')$ also do not exceed 1, with

$$\beta_s + \gamma_{t+1-s} = \widetilde{h} \qquad \sum_{s=1}^{t}(\beta_s + \gamma_s) = \widetilde{h}t \tag{1.15a}$$

in the first case and

$$\widetilde{h} - \beta_{t-s+1} \geqslant \gamma_s \geqslant \widetilde{h} - \beta_{t-s} \geqslant \gamma_{s+1} \qquad \sum_{s=1}^{t}(\beta_s + \gamma_s) = \widetilde{h}(t-1) + h \tag{1.15b}$$

in the second. We may interchange the rows and columns of α', β and γ, and use relation (1.14*b*) in the third case.

For $a \leqslant 0$ or $b \leqslant 0$ integers, the number of columns in β is restricted: $\beta_s \leqslant \beta_1 \leqslant -a$ or $-b$, as well as for $c+d \leqslant 0$ or $c+e \leqslant 0$ integers, the partition γ is restricted in (1.11*b*): $\gamma_1 \leqslant -c-d$ or $-c-e$. Otherwise, for a or b positive integers, the number of rows in β is restricted: $\beta_{s'} = 0$, if $s' > a$ or b, as well as for $c+d$ or $c+e$ positive integers, $\gamma_{s'} = 0$, if $s' > c+d$ or $c+e$. When these restrictions exceed those caused by conditions (1.15*a*) or (1.15*b*), some common factors (appearing in all non-vanishing terms of (1.11*b*)) may be carried out before the expression, in order to write the reduction formulae of A_λ functions with shifted values of α_s and a, b, d, e, c. For specified values of a, b, d, or e, the roots of these common factors give the sets of zeros of the functions A_α, distributed as the SU(3) weight space of zeros [13] or the truncated weight space of zeros [1, 16], respectively.

For special partitions $\dot{\alpha} = [\widetilde{h}^t]$, the function $A_\alpha\binom{a,b,d,e}{c}$ may be expressed (cf equation (1.25) of [14]) in a more symmetric form:

$$\begin{aligned} A_{\dot{\alpha}}\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} = \sum_{\beta}[M(\beta)M(\bar{\beta})]^{-1}\prod_{s=1}^{t}(-1)^{\beta_s}(a-s+1)_{\beta_s}(a+c-s+1)_{\beta_s} \\ \times(a+b+d+e+2c+\widetilde{h}-t-s+1)_{\beta_s}(a+b+c+\beta_s-s+1)_{\alpha_s-\beta_s} \\ \times(a+d+c+\beta_s-s+1)_{\alpha_s-\beta_s}(a+e+c+\beta_s-s+1)_{\alpha_s-\beta_s} \end{aligned} \tag{1.16}$$

where

$$\bar{\beta}_s = \widetilde{h} - \beta_{t-s+1}.$$

Of course, equations (1.11) and (1.16) simplify considerably for $t = 1$ (turning into the ${}_4F_3(1)$ hypergeometric series) and, in particular, $A_{[0]}\binom{a,b,d,e}{c} = 1$.

2. Orthogonalization coefficients for the paracanonical and pseudocanonical isofactors of SU(3)

In this section, we present the explicit polynomial expressions of the boundary orthonormal isofactors of SU(3) ⊃ SO(3), which correspond to the paracanonical and pseudocanonical labelling schemes and form the triangular orthogonalization matrices of the Gram–Schmidt process for the non-orthogonal isofactors satisfying simple boundary conditions [15]. Presented inverse matrices are formed by OCs of the standard (minimal) bilinear combinations of isofactors. Although we cannot avoid the Gram–Schmidt process for these versions of the SU(3) coproduct splitting (contrary to the canonical splitting [27]), the structure of the numerator–denominator polynomials is simpler than for the canonical splitting of Louck *et al* [13].

We use in this section the same notation for the irreps and their basis states as in [15, 16], with $(a\ b)$ for the mixed tensor irreps where $a = m_{13} - m_{23}$, $b = m_{23} - m_{33}$ and $[m_{13}, m_{23}, m_{33}]$ is a Young frame (partition). The basis states are labelled by the hypercharge $y = m_{12} + m_{22} - \frac{2}{3}(m_{13} + m_{23} + m_{33})$, the isospin $i = \frac{1}{2}(m_{12} - m_{22})$, and its projection $i_z = m_{11} - \frac{1}{2}(m_{12} + m_{22})$, where the integers m_{ij} are the Gelfand–Tsetlin parameters. Sometimes the parameter

$$z = \tfrac{1}{3}(b-a) - \tfrac{1}{2}y = m_{23} - \tfrac{1}{2}(m_{12} + m_{22}) \tag{2.1}$$

is more convenient than y, because

$$i \pm z \geqslant 0 \qquad a + z - i \geqslant 0 \qquad b - z - i \geqslant 0 \tag{2.2}$$

are integers. In the case of coupling $(a'b') \otimes (a''b'')$ to $(a\ b)$

$$z = z' + z'' + v \qquad \text{where} \quad v = \tfrac{1}{3}(a' - b' + a'' - b'' - a + b) \tag{2.3}$$

is an integer. We also use the following notation for the parameters of the highest- and lowest-weight states:

$$y_0' = \tfrac{1}{3}(a' + 2b') \quad i_0' = -z_0' = \tfrac{1}{2}a' \qquad \overline{y}_0'' = -\tfrac{1}{3}(2\overline{a}'' + \overline{b}'') \quad \overline{i}_0'' = \overline{z}_0'' = \tfrac{1}{2}b''. \tag{2.4}$$

The boundary paracanonical isofactors and inverse orthogonalization coefficients $Q_{j,\widetilde{I}}$ with label $\widetilde{I}$ of repeating irreps satisfy the conditions of biorthogonality

$$\sum_{\widetilde{I}} Q_{j,\widetilde{I}}(a'b'y_0'i_0'a''b''\overline{y}_0''\overline{i}_0''||a\,b\,\widetilde{y}i;\widetilde{I}) = \delta_{j,i} \tag{2.5a}$$

$$\sum_{i}(a'b'y_0'i_0'a''b''\overline{y}_0''\overline{i}_0''||a\,b\,\widetilde{y}i;\,\widetilde{I})Q_{i,\widetilde{J}} = \delta_{\widetilde{I},\widetilde{J}}. \tag{2.5b}$$

For the chosen version [15, 16] of the paracanonical coupling the multiplicity label $\widetilde{I}$ (the intrinsic isospin of the Gelfand–Weyl–Biedenharn pattern) satisfies the conditions

$$\begin{array}{lll} \widetilde{I} \pm \widetilde{z} \geqslant 0 & a + \widetilde{z} - \widetilde{I} \geqslant 0 & b - \widetilde{z} - \widetilde{I} \geqslant 0 \\ \widetilde{I} \pm \widetilde{i}_z \geqslant 0 & i_0' + \overline{i}_0'' - \widetilde{I} \geqslant 0 & \widetilde{I} \geqslant B \end{array} \tag{2.6a}$$

where

$$B = \tfrac{1}{2}(a + b - b' - a'' + |v|) \qquad \widetilde{i}_z = \tfrac{1}{2}(a' - b'') \qquad \widetilde{z} = \tfrac{1}{2}(b'' - a') + v. \tag{2.6b}$$

Expressions (2.1) and (4.1) of [16] for the boundary paracanonical isofactors and corresponding inverse orthogonalization coefficients $Q_{j,\widetilde{I}}$ with the fixed values of $b - 2\widetilde{z} \equiv b' - a'' + a + v$, $i - \widetilde{z}$, and $b - \widetilde{z} - \widetilde{I}$ may be written as follows:

$$(a'b'y_0'i_0', a''b''\overline{y}_0''\overline{i}_0''||a\ b\ \widetilde{y}i;\,\widetilde{I}) = \frac{(-1)^{\widetilde{I}-i}\ \mathcal{L}\,\mathbf{K}_{\widetilde{I},i}\ g_{\widetilde{I},i}}{(\widetilde{I}-i)!\,[g_{\widetilde{I},\widetilde{I}}\ g_{\widetilde{I}+1,\widetilde{I}+1}]^{1/2}} \tag{2.7a}$$

$$Q_{j,\widetilde{I}}^{(a'b',a''b'';ab)} = \frac{(2j+1)\,\mathbf{K}_{j,\widetilde{I}}\,g^{j,\widetilde{I}}}{(j-\widetilde{I})!(j+\widetilde{I}+1)\,\mathcal{L}\,[g_{\widetilde{I},\widetilde{I}}\,g_{\widetilde{I}+1,\widetilde{I}+1}]^{1/2}}. \tag{2.7b}$$

Here

$$\mathcal{L} = \left[\frac{(a+1)(b+1)(a+b+2)b'^{(b-\widetilde{z}-\widetilde{I})}(a'+b'+1)^{(b-\widetilde{z}-\widetilde{I})}(b-\widetilde{z}+i+1)^{(b-\widetilde{z}-\widetilde{I})}}{(a''+1)_{\widetilde{I}-\widetilde{z}+1}(a''+b''+2)_{\widetilde{I}-\widetilde{z}+1}(i+\widetilde{z}+1)_{\widetilde{I}-\widetilde{z}+1}}\right]^{1/2} \tag{2.8}$$

includes the linear numerator and denominator factors, appearing together with their counterparts from the symmetry properties and decomposition of the Gram determinants and eliminating the superfluous matrix elements. The factor

$$\mathbf{K}_{j,i} = [(j+\widetilde{i}_z)^{(j-i)}(j-\widetilde{i}_z)^{(j-i)}(i'_0+\bar{i}''_0-i)^{(j-i)}(i'_0+\bar{i}''_0+j+1)^{(j-i)}(j-\widetilde{z})^{(j-i)} \times (a+\widetilde{z}-i)^{(j-i)}(a+\widetilde{z}+j+1)^{(j-i)}(b-\widetilde{z}-i)^{(j-i)}]^{1/2} \tag{2.9}$$

(with 0 for $j < i$) appears as an analogue of pattern calculus factors. The numerator–denominator polynomials may be expressed as

$$g_{\widetilde{I},i} = M(\alpha)A_\alpha\begin{pmatrix}\mathrm{a, b, d, e}\\ \mathrm{c}\end{pmatrix} \qquad g^{j,\widetilde{I}} = M(\alpha')A_{\alpha'}\begin{pmatrix}\mathrm{a, b, d, e}\\ \mathrm{c}-1\end{pmatrix} \tag{2.10}$$

where

$$\mathrm{a} = b'+v+1 \qquad \mathrm{b} = a'+a''+b-v+3 \qquad \mathrm{d} = a''+b-v+2$$

$$\mathrm{e} = b'+b''+v+2 \qquad \mathrm{c} = v-\widetilde{z}-\widetilde{I}-a'-a''-b'-3$$

and

$$\alpha = [\widetilde{h}^{t-1}, h] \qquad \alpha' = [(\widetilde{h}+1)^{t-l-1}\widetilde{h}^{l}] \qquad t = b-\widetilde{z}-\widetilde{I}+1$$

$$\widetilde{h} = \widetilde{I}-\widetilde{z} \qquad h = i-\widetilde{z} \qquad l = j-\widetilde{I}.$$

In particular

$$g_{b-\widetilde{z}+1,b-\widetilde{z}+1} = g_{\widetilde{z},\widetilde{z}} = g_{b-\widetilde{z},\widetilde{z}} = g^{b-\widetilde{z},b-\widetilde{z}} = g^{\widetilde{z}-1,\widetilde{z}-1} = g^{b-\widetilde{z},\widetilde{z}} = 1$$

$$g^{\widetilde{I},\widetilde{I}} = g_{\widetilde{I}+1,\widetilde{I}+1} \qquad g^{\widetilde{I}+1,\widetilde{I}} = g_{\widetilde{I}+1,\widetilde{I}}.$$

We modify the formal structure of the boundary pseudocanonical isofactors slightly (cf equations (4.11) of [15]) and express them (for fixed $b-2\widetilde{z}$, $i-\widetilde{z}$, and $b-\widetilde{z}-\widetilde{j}$) together with the corresponding inverse orthogonalization coefficients as follows:

$$(a'b'y'_0i'_0a''b''\bar{y}''_0\bar{i}''_0||a\ b\ \widetilde{y}i;\widetilde{j}) = \frac{(-1)^{i-\widetilde{j}}\,\overline{\mathcal{L}}\,\mathbf{K}_{i,\widetilde{j}}\,\bar{g}_{i,\widetilde{j}}}{(i-\widetilde{j})!\,[\bar{g}_{\widetilde{j},\widetilde{j}}\,\bar{g}_{\widetilde{j}-1,\widetilde{j}-1}]^{1/2}} \tag{2.11a}$$

$$\overline{Q}^{(a'b',a''b'';ab)}_{\widetilde{j},i} = \frac{(2i+1)\,\mathbf{K}_{\widetilde{j},i}\,\bar{g}^{\widetilde{j},i}}{(\widetilde{j}-i)!\,(\widetilde{j}+i+1)\,\overline{\mathcal{L}}\,[\bar{g}_{\widetilde{j},\widetilde{j}}\,\bar{g}_{\widetilde{j}-1,\widetilde{j}-1}]^{1/2}} \tag{2.11b}$$

where

$$\overline{\mathcal{L}} = \left[\frac{(a+1)(b+1)(a+b+2)\,b'^{(\widetilde{j}-\widetilde{z})}(a'+b'+1)^{(\widetilde{j}-\widetilde{z})}(i+\widetilde{z}+1)_{\widetilde{j}-\widetilde{z}}}{(a''+1)_{b-\widetilde{z}-\widetilde{j}+1}(a''+b''+2)_{b-\widetilde{z}-\widetilde{j}+1}(b-\widetilde{z}+i+1)^{(b-\widetilde{z}-\widetilde{j}+1)}}\right]^{1/2}. \tag{2.12}$$

We see that equations (2.11a) and (2.11b) may be obtained from (2.7a) and (2.7b), respectively, by means of the substitutions

$$\begin{aligned} &a \to a+b+1 \qquad b \to -b-2 \qquad v \to -b+v-1 \\ &\widetilde{z} \to \widetilde{z}-b-1 \qquad i \to -i-1 \qquad \widetilde{I} \to -\widetilde{j}-1 \end{aligned} \tag{2.13}$$

which leaves the eigenvalues of the SU(3) and SU(2) Casimir operators invariant. The parameters a', b', a'', b'', $a+\tilde{z}$, $b-2\tilde{z}$, $a+v$ are also left unchanged. The same substitutions into (2.10) allows us to write the expressions for the numerator–denominator polynomials

$$\overline{g}_{i,\tilde{j}} = (-1)^{\sum_s \overline{\alpha}_s} M(\overline{\alpha}) A_{\overline{\alpha}} \begin{pmatrix} \mathrm{a}', \mathrm{b}', \mathrm{d}', \mathrm{e}' \\ \mathrm{c}' \end{pmatrix} \tag{2.14a}$$

$$\overline{g}^{\tilde{j},i} = (-1)^{\sum_s \overline{\alpha}'_s} M(\overline{\alpha}') A_{\overline{\alpha}'} \begin{pmatrix} \mathrm{a}', \mathrm{b}', \mathrm{d}', \mathrm{e}' \\ \mathrm{c}' - 1 \end{pmatrix} \tag{2.14b}$$

where

$$\mathrm{a}' = b' - b + v \qquad \mathrm{b}' = a' + a'' - v + 2 \qquad \mathrm{d}' = a'' - v + 1$$

$$\mathrm{e}' = b' + b'' - b + v + 1 \qquad \mathrm{c}' = v + \tilde{j} - \tilde{z} - a' - a'' - b' - 2$$

and

$$\overline{\alpha} = [\tilde{h}^{t-1}, h] \qquad t = \tilde{j} - \tilde{z} + 1 \qquad \tilde{h} = b - \tilde{z} - \tilde{j}$$

$$h = b - \tilde{z} - i \qquad \overline{\alpha}'^{*} = [(\tilde{j} - \tilde{z})^{b-\tilde{z}-\tilde{j}}, i - \tilde{z}].$$

The denominator polynomials $\overline{g}_{\tilde{j},\tilde{j}}$ and $\overline{g}^{\tilde{j},\tilde{j}}$ may accept only non-negative values. In particular

$$\overline{g}_{b-\tilde{z},b-\tilde{z}} = \overline{g}_{\tilde{z}-1,\tilde{z}-1} = \overline{g}_{b-\tilde{z},\tilde{z}} = \overline{g}^{\,b-\tilde{z}+1,b-\tilde{z}+1} = \overline{g}^{\,\tilde{z},\tilde{z}} = \overline{g}^{\,b-\tilde{z},\tilde{z}} = 1$$

$$\overline{g}^{\tilde{j},\tilde{j}} = \overline{g}_{\tilde{j}-1,\tilde{j}-1} \qquad \overline{g}^{\,\tilde{j},\tilde{j}-1} = \overline{g}_{\tilde{j},\tilde{j}-1}.$$

We can check equations (2.11*a*), (2.11*b*) and (2.14) for extreme values of $\tilde{j}$ with the corresponding expressions of the overlaps [15], but the substitutions (2.13) allow us to avoid the necessity of reconsidering for the polynomials $\overline{g}_{i,\tilde{j}}$ and $\overline{g}^{\tilde{j},i}$ the arguments about their polynomial structure, symmetries, reduction formulae and distribution of zeros, used in [16] in the case of the polynomials $g_{\tilde{I},i}$.

In the same way as the corresponding function (1.15) of [16], the function

$$\frac{(a'b'y_0'i_0'a''b''\overline{y}_0''\overline{i}_0''||a\,b\,\tilde{y}i;\,\tilde{j})}{[(a+1)(b+1)(a+b+2)(a'+1)(b''+1)M(a'+b',b')M(a''+b'',a'')]^{1/2}} \tag{2.15}$$

is invariant under 24 transformations of array (1.13) of [16] and, in particular, with respect to the substitutions

$$\begin{aligned} &a \to b' - a'' + a + v \qquad b \to a'' - b' + b - v \\ &b' \to a'' - v \qquad a'' \to b' + v \end{aligned} \tag{2.16}$$

(and $a + \tilde{z} \leftrightarrow b - \tilde{z}$, with fixed a', b'', $v, \tilde{z}$ and $\tilde{i}_z$). This last Regge-type symmetry is also very important for the results presented in the next section. Recall that the general SU(3) pseudocanonical isofactors may be expanded in terms of the minimal biorthogonal systems [15] (cf equation (1.11) of [16]).

3. Explicit orthogonalization of the projected bases of two-parametric irreps

The repeating irreps $[L_1L_2]$ of SO(n) in the projected states of the representation $(\lambda\nu\dot{0})$ of SU(n) are distinguished by the intrinsic multiplicity label k:

$$\tfrac{1}{2}(\Delta_0+\delta_0) \leqslant k \leqslant \tfrac{1}{2}\min(\lambda,\ L_1-L_2) \tag{3.1}$$

where $\Delta_0 = 0$ or 1, $\delta_0 = 0$ or 1; $\nu - L_2 - \Delta_0$ and $\lambda+\nu-L_2-\delta_0$ are even integers and $\frac{1}{2}\lambda - k$ is an integer.

When the linearly independent states are chosen as in [1, 3–8] from above with $k \geqslant \frac{1}{2}(L_1-\Delta_0-\nu)$, the orthogonalization coefficients from [1]† may be presented in the following more compact forms:

$$\widetilde{B}^{(\lambda\nu,L_1L_2)_n}_{\kappa,k} = \frac{(2k)^{(\Delta_0+\delta_0)/2}\,\mathbf{B}^{[\lambda\nu,L_1L_2]n}_{\kappa,k}\,\mathbf{g}_{\kappa,k}}{(\kappa-k)!(\kappa+k)!\left[\mathbf{g}_{\kappa,\kappa}\,\mathbf{g}_{\kappa+1,\kappa+1}\right]^{1/2}} \tag{3.2a}$$

$$B^{(\lambda\nu,L_1L_2)_n}_{k,\kappa} = \frac{(-1)^{k-\kappa}\,(2k)^{1-(\Delta_0+\delta_0)/2}\,(\kappa+k-1)!\,\mathbf{g}^{k,\kappa}}{(k-\kappa)!\left[\mathbf{g}_{\kappa,\kappa}\ \mathbf{g}_{\kappa+1,\kappa+1}\right]^{1/2}\mathbf{B}^{[\lambda\nu,L_1L_2]n}_{\kappa,k}}. \tag{3.2b}$$

Here non-negative even integers in the double factorial $(\nu-L_1-\Delta_0+2\kappa)!!$ (appearing under the square root sign together with their counterparts from the decomposition of the Gram determinants in accordance with their symmetry properties) in the numerator of

$$\mathbf{B}^{[\lambda\nu,L_1L_2]n}_{\kappa,k} = \frac{2^{-(\lambda+n-3)/2-\nu}\,N(\lambda\nu;k[L_1L_2]_n)}{[(\nu-L_1-\Delta_0+2\kappa)!!(\nu+L_2+2\tau-\Delta_0+2\kappa+n-2)!!]^{1/2}} \times\frac{R(k[L_1L_2]\kappa;\sigma\tau)_n}{[(\nu-L_1+2\tau-\delta_0+2\kappa+1)!!(\nu+L_2-\delta_0+2\kappa+n-3)!!]^{1/2}} \tag{3.3}$$

restrict the parameters and eliminate the superfluous matrix elements of OCs. The factor

$$R(k[L_1L_2]\kappa;\sigma\tau)_n = 2^{2k-\Delta_0-\delta_0-\sigma}\left[\frac{(\sigma+k)!(\tau+\kappa)!(\sigma-k)!(\tau-k)!}{(\tau+k)!(\sigma-\kappa)!(\tau-\kappa)!}\right.\times\frac{(L_1+L_2+2\kappa+n-4)!!(L_1+L_2-2k+n-4)!!}{(L_1+L_2+2k+n-4)!!(L_1+L_2-2\kappa+n-4)!!}\times\left.\frac{[\kappa-(\Delta_0+\delta_0)/2]!(2\kappa-\delta_0+\Delta_0-1)!!(2\kappa+\delta_0-\Delta_0-1)!!}{(2k+\delta_0-\Delta_0-1)!!(2k-\delta_0+\Delta_0-1)!!}\right]^{1/2} \tag{3.4}$$

appeared in accordance with the factors under the square root of the overlaps of non-orthogonal states. The notation

$$\sigma = \tfrac{1}{2}\min(L_1-L_2,\ \lambda) \qquad \tau = \tfrac{1}{2}\max(\lambda,\ L_1-L_2) \tag{3.5}$$

permitted to merge in (3.2a), (3.2b) both versions, presented as equations (2.3), (2.4) and (2.13), (2.14) of [1], respectively; $N(\lambda\nu;k[L_1L_2]_n)$ is defined as (1.4). For $k=\Delta_0=\delta_0=0$ in (3.2a), the indeterminacy $(2k)^{(\Delta_0+\delta_0)/2} = 1$, as well as for $k=\kappa=0$ in (3.2b), $(2k)(\kappa+k-1)! = 1$.

† Note that the parameters $k+\frac{1}{2}\lambda+\nu$ should be written jointly in (1.2) of [1], $\mathbf{g}_{\kappa\kappa}$ should appear instead of $\mathbf{g}_{\kappa k}$ in the denominators of (2.3) and (2.13), as well as $\mathbf{g}^{k\kappa}$ instead of $g^{k\kappa}$ in the numerator of (2.4); in the second row of the r.h.s. of (2.8), $-2+s]^{(\alpha_s')}$ should be replaced by $-1+\kappa+s]^{(\alpha_s')}$ and $-\delta_0+1$ should be replaced by $-\delta_0-1$ in the last row of (2.27b).

The numerator–denominator polynomials (functions) in (3.2*a*) and (3.2*b*) may be expressed as (2.7), (2.8), (2.16), (2.17) of [1] and are combined as follows:

$$\mathbf{g}_{\kappa,k} = M(\alpha)A_\alpha\begin{pmatrix} a,b,d,e \\ c \end{pmatrix} \qquad \mathbf{g}^{k,\kappa} = M(\alpha')A_{\alpha'}\begin{pmatrix} a,b,d,e \\ c-1 \end{pmatrix} \tag{3.6}$$

where

$$\begin{aligned} \alpha &= [(\kappa - \tfrac{1}{2}(\Delta_0+\delta_0))^{\sigma-\kappa}, k - \tfrac{1}{2}(\Delta_0+\delta_0)] \\ \alpha' &= [(\kappa - \tfrac{1}{2}(\Delta_0+\delta_0)+1)^{\sigma-k}, (\kappa - \tfrac{1}{2}(\Delta_0+\delta_0))^{k-\kappa}] \end{aligned} \tag{3.7}$$

$$\begin{aligned} & a = \tfrac{1}{2}(\lambda+\nu+L_1+\Delta_0+n-1) \qquad b = \sigma + \tfrac{1}{2}(\nu - L_1 + \Delta_0 + 1) \\ & c = -\tfrac{1}{2}(\lambda+2\nu+n-1) - \kappa \qquad d = \sigma + \tfrac{1}{2}(\nu + L_2 + \delta_0 + n - 2) \\ & e = \tfrac{1}{2}(\lambda+\nu-L_2+\delta_0)+1 \end{aligned} \tag{3.8}$$

and, in particular,

$$\begin{aligned} & \mathbf{g}_{\sigma+1,\sigma+1} = \mathbf{g}_{(\Delta_0+\delta_0)/2,(\Delta_0+\delta_0)/2} = 1 \\ & \mathbf{g}^{\kappa,\kappa} = \mathbf{g}_{\kappa+1,\kappa+1} \qquad \mathbf{g}^{\kappa+1,\kappa} = \mathbf{g}_{\kappa+1,\kappa}. \end{aligned} \tag{3.9}$$

Of course, the partitions (3.7) are formed by integers, but some parameters a, b, c, d, e, or their linear combinations appearing in expressions (1.11*b*) and (1.16) of polynomial A_α may be half-integers.

At last we may present the main result of this paper. Following Vergados [17] and Tolstoy [18], and in contrast to [1], we choose the linearly dependent states with parameters $k > \frac{1}{2}\min(\lambda+\nu-L_1+\delta_0,\ \nu-L_2+\delta_0)$. In this case the multiplicity label $\overline{\kappa}$ of the generalized orthonormal Vergados–Elliott–Draayer states satisfy conditions:

$$\tfrac{1}{2}(\Delta_0+\delta_0) \leqslant \overline{\kappa} \leqslant \tfrac{1}{2}\min(\lambda,\ L_1-L_2,\ \lambda+\nu-L_1+\delta_0,\ \nu-L_2+\delta_0). \tag{3.10}$$

Using arguments similar to those used in [1] (cf also [2, 15, 16]) about the symmetry and the linear factors of the OCs in the numerator and denominator under the square root, as well as about the polynomial structure and zeros (roots) of the numerator–denominator polynomials and the total OCs, we derived the following expressions for the orthogonalization coefficients $\widetilde{\mathcal{O}}^{(\lambda\nu,L_1L_2)_n}_{k,\overline{\kappa}}$ and $\mathcal{O}^{(\lambda\nu,L_1L_2)_n}_{\overline{\kappa},k}$:

$$\widetilde{\mathcal{O}}^{(\lambda\nu,L_1L_2)_n}_{k,\overline{\kappa}} = \frac{(1+\delta_{k,0})(\overline{\kappa}+k-1)!\,(2k)^{1-(\Delta_0+\delta_0)/2}\,\overline{\mathbf{g}}_{k,\overline{\kappa}}\,\overline{\mathbf{B}}^{[\lambda\nu,L_1L_2]n}_{k,\overline{\kappa}}}{(k-\overline{\kappa})!\left[\overline{\mathbf{g}}_{\overline{\kappa},\overline{\kappa}}\,\overline{\mathbf{g}}_{\overline{\kappa}-1,\overline{\kappa}-1}\right]^{1/2}} \tag{3.11a}$$

$$\mathcal{O}^{(\lambda\nu,L_1L_2)_n}_{\overline{\kappa},k} = \frac{(-1)^{\overline{\kappa}-k}\,(1+\delta_{k,0})^{-1}\,(2k)^{(\Delta_0+\delta_0)/2}\,\overline{\mathbf{g}}^{\overline{\kappa},k}}{(\overline{\kappa}-k)!(\overline{\kappa}+k)!\left[\overline{\mathbf{g}}_{\overline{\kappa},\overline{\kappa}}\,\overline{\mathbf{g}}_{\overline{\kappa}-1,\overline{\kappa}-1}\right]^{1/2}\overline{\mathbf{B}}^{[\lambda\nu,L_1L_2]n}_{k,\overline{\kappa}}}. \tag{3.11b}$$

Here non-negative even integers in the double factorial $(\nu - L_1 + 2\sigma + \delta_0 - 2\overline{\kappa})!!$ (again appearing together with their partners, corresponding to the symmetry properties of the overlaps) in the numerator of

$$\begin{aligned} \overline{\mathbf{B}}^{[\lambda\nu,L_1L_2]n}_{k,\overline{\kappa}} = {} & 2^{-(\lambda+n-1)/2-\nu}\,\frac{N(\lambda\nu;k[L_1L_2]_n)}{R(k[L_1L_2]\overline{\kappa};\sigma\tau)_n}\,[(\nu-L_1+2\sigma+\delta_0-2\overline{\kappa})!! \\ & \times(\lambda+\nu+L_1+\delta_0-2\overline{\kappa}+n-2)!!(\lambda+\nu-L_2+\Delta_0+1-2\overline{\kappa})!! \\ & \times(\nu+L_2+2\sigma+\Delta_0-2\overline{\kappa}+n-3)!!]^{-1/2} \end{aligned} \tag{3.12}$$

restrict the parameters and eliminate the superfluous matrix elements of the OCs. In (3.12) the same auxiliary notation for the renormalization factor (1.4), the ratio (3.4) of the linear (pattern calculus) factors under the square root sign and parameters σ, τ (3.5) for merging of the different solutions are used. Again for $k = \kappa = \Delta_0 = \delta_0 = 0$ in (3.11*a*), the indeterminacy $(\kappa + k - 1)!(2k)^{1-(\Delta_0+\delta_0)/2} = 1$ and in (3.11*b*), $(2k)^{(\Delta_0+\delta_0)/2} = 1$. (The additional factors $(1 + \delta_{k,0})$ and $(1 + \delta_{k,0})^{-1}$ will be explained later). The numerator–denominator polynomials (functions) are expressed in (3.11*a*) and (3.11*b*) as follows:

$$\overline{\mathbf{g}}_{k,\overline{\kappa}} = (-1)^{\sum_s \beta_s} M(\beta) A_\beta \begin{pmatrix} a', b', d', e' \\ c' \end{pmatrix} \tag{3.13a}$$

$$\overline{\mathbf{g}}^{\overline{\kappa},k} = (-1)^{\sum_s \beta'_s} M(\beta') A_{\beta'} \begin{pmatrix} a', b', d', e' \\ c' - 1 \end{pmatrix} \tag{3.13b}$$

where

$$\begin{aligned} \beta &= [(\sigma - \overline{\kappa})^{\overline{\kappa}-(\Delta_0+\delta_0)/2}, \sigma - k] \\ \beta' &= [(\sigma - \overline{\kappa} + 1)^{k-(\Delta_0+\delta_0)/2}, (\sigma - \overline{\kappa})^{\overline{\kappa}-k}] \end{aligned} \tag{3.14}$$

$$\begin{aligned} a' &= \tfrac{1}{2}(\nu + L_2 - \Delta_0 + n - 2) \qquad & b' &= \tau + \tfrac{1}{2}(\nu - L_1 - \Delta_0) + 1 \\ c' &= -\tfrac{1}{2}(\lambda + 2\nu + n - 1) + \overline{\kappa} \qquad & d' &= \tau + \tfrac{1}{2}(\nu + L_2 - \delta_0 + n - 1) \\ e' &= \tfrac{1}{2}(\nu - L_1 - \delta_0 + 1) \end{aligned} \tag{3.15}$$

and, in particular, for $\beta = 0$ or $\beta' = 0$

$$\begin{aligned} &\overline{\mathbf{g}}_{\sigma,\sigma} = \overline{\mathbf{g}}^{(\Delta_0+\delta_0)/2,(\Delta_0+\delta_0)/2} = \overline{\mathbf{g}}^{\sigma,(\Delta_0+\delta_0)/2} = 1 \\ &\overline{\mathbf{g}}^{\overline{\kappa},\overline{\kappa}} = \overline{\mathbf{g}}_{\overline{\kappa}-1,\overline{\kappa}-1} \qquad \overline{\mathbf{g}}^{\overline{\kappa},\overline{\kappa}-1} = \overline{\mathbf{g}}_{\overline{\kappa},\overline{\kappa}-1}. \end{aligned} \tag{3.16}$$

Again, some parameters a', b', c', d', e' of the polynomials A_α may be half-integers. Of course, the denominator polynomials $\overline{\mathbf{g}}_{\overline{\kappa},\overline{\kappa}}$ accept only non-negative values.

Equations (3.11*a*), (3.11*b*) and (3.13) were proved preliminarily for extreme values of $\overline{\kappa} = \frac{1}{2}(\Delta_0 + \delta_0)$ and $\overline{\kappa} = \frac{1}{2}(L_1 - L_2)$. The boundary OCs may be expressed in terms of the overlaps as follows:

$$\widetilde{\mathcal{O}}^{(\lambda\nu,L_1L_2)_n}_{k,(\Delta_0+\delta_0)/2} = \frac{\langle [E]\,k \,\big|[E]\,\frac{1}{2}(\Delta_0+\delta_0)\rangle}{\big\langle [E]\,\frac{1}{2}(\Delta_0+\delta_0) \,\big|[E]\,\frac{1}{2}(\Delta_0+\delta_0)\big\rangle^{1/2}} \tag{3.17a}$$

$$\mathcal{O}^{(\lambda\nu,L_1L_2)_n}_{(L_1-L_2)/2,k} = \frac{\big\langle [\overline{E}]\,\frac{1}{2}(L_1-L_2) \,\big|[\overline{E}]\,k\big\rangle}{\big\langle [\overline{E}]\,\frac{1}{2}(L_1-L_2) \,\big|[\overline{E}]\,\frac{1}{2}(L_1-L_2)\big\rangle^{1/2}}\,. \tag{3.17b}$$

For this purpose the overlap

$$\langle [E]\,k \,\big|[E]\,\tfrac{1}{2}(\Delta_0+\delta_0)\rangle \equiv \left\langle \begin{matrix} (\lambda\nu\dot{0})_{[E]} \\ k[L_1L_2]_n \\ \cdots \end{matrix} \,\middle|\, \begin{matrix} (\lambda\nu\dot{0})_{[E]} \\ \frac{1}{2}(\Delta_0+\delta_0)\,[L_1L_2]_n \\ \cdots \end{matrix} \right\rangle \tag{3.18a}$$

should be expressed using the corrected formula (5.4) of [8], together with our equation (1.3*b*). Since the summation parameter $x = 0$ is fixed and the sum over l_2 is equivalent to the summable Saalschutzian [25, 26] (balanced) ${}_3F_2(1)$ hypergeometric series, the last sum over l_0 in the final expression for (3.18*a*) corresponds to the balanced ${}_4F_3(1)$ hypergeometric series in a related to (1.16) form in accordance with the first expression (3.13) for $\bar{\mathbf{g}}_{k,(\Delta_0+\delta_0)/2}$. Otherwise, the overlap

$$\langle [\overline{E}]\,\tfrac{1}{2}(L_1-L_2)\;|[\overline{E}]\,k\rangle \equiv \left\langle \begin{array}{c} (\lambda\nu\dot{0})_{[\overline{E}]} \\ \frac{1}{2}(L_1-L_2)\,[L_1L_2]_n \\ \cdots \end{array} \right| \left. \begin{array}{c} (\lambda\nu\dot{0})_{[\overline{E}]} \\ k\,[L_1L_2]_n \\ \cdots \end{array} \right\rangle \tag{3.18b}$$

for $k' = \frac{1}{2}(L_1-L_2) \leqslant \frac{1}{2}\lambda$ may be expressed by means of equation (5.7) of [19], together with our formula (1.3*b*). In this case the expansion over $\underline{k}' = k'$ is trivial, the summation parameter $\bar{l}_2$ is restricted as $\bar{l}_2 = L_2 + \Delta_0 + 2z$ and the sum over z is of the type, corresponding to the summable balanced ${}_3F_2$ series, as well as the afterwards obtained sum over y. Hence, the last sum over t of the final expression for (3.18*b*) corresponds to the balanced ${}_4F_3(1)$ hypergeometric series, which again may be presented in the form related to (1.16) in accordance with expression (3.13*b*) for $\bar{\mathbf{g}}^{(L_1-L_2)/2,k}$, taking into account relation (1.14*a*). The factor $(1+\delta_{k,0})^{-1}$ in (3.11*b*) and in the expression for (3.18*b*) appearing from (5.7) of [19] can be replaced by $\frac{1}{2}$ when the orthonormal states are expanded in terms of the linearly dependent projected states with positive and negative k.

Up to the $\bar{\kappa}$-independent renormalization factors, determined completely by the strictly derived formulae (3.17*a*) and (3.17*b*), the orthonormalization coefficients (3.11*a*) and (3.11*b*) may be obtained by an analytical continuation of the OCs (2.11*a*) and (2.11*b*) of the pseudocanonical isofactors, in a similar manner to the way the OCs (3.2*a*) and (3.2*b*) were obtained by means of the analytical continuation [1] of the OCs (2.7*a*) and (2.7*b*) of the paracanonical isofactors [15, 16]. For fixed L_1-L_2, the parameters of (2.11*a*) and (2.11*b*) should be replaced by means of the same substitutions as in [1], namely

$$\begin{array}{l} i \to k-\frac{1}{2} \qquad \widetilde{j} \to \bar{\kappa}-\frac{1}{2} \qquad \widetilde{z} \to \frac{1}{2}(\Delta_0+\delta_0-1) \\ \widetilde{i}_z \to \frac{1}{2}(\delta_0-\Delta_0) \qquad v \to \delta_0-\frac{1}{2} \\ a' \to -\frac{1}{2}(L_1+L_2+\Delta_0-\delta_0+n-1) \qquad b' \to \frac{1}{2}(\nu+L_1-\delta_0+n-3) \\ a'' \to \frac{1}{2}(\lambda+\nu+L_2+\delta_0+n)-2 \qquad b'' \to -\frac{1}{2}(L_1+L_2-\Delta_0+\delta_0+n-1) \\ a \to \frac{1}{2}(\lambda-\Delta_0-\delta_0) \qquad b \to \frac{1}{2}(L_1-L_2+\Delta_0+\delta_0)-1 \end{array} \tag{3.19}$$

in accordance with the correlated symmetries and restricting properties of the boundary SU(3) $\supset$ U(2) and SU(n) $\supset$O(n) isofactors and the corresponding overlap coefficients, visually represented as array (1.13) of [16] and array (2.23) of [1].

Similarly, for fixed λ, expressions (3.11*a*) and (3.11*b*) may be obtained by means of the substitutions (3.19) applied after (2.16). Finally, we presented the both couples of solutions (for $L_1-L_2 \leqslant \lambda$ and for $L_1-L_2 \geqslant \lambda$) in the unified forms as (3.11*a*) and (3.11*b*) (and the analogous results of [1] as (3.2*a*) and (3.2*b*)). It was convenient also to replace in (3.2*a*), (3.2*b*), (3.11*a*) and (3.11*b*) some ratios of factorials of the type (1.10*b*) by elementary powers of $(2k)$ (with the exponents $(\Delta_0+\delta_0)/2$ or $1-(\Delta_0+\delta_0)/2$). However, we see that neither the inverse reconstruction of (2.14) and (2.17) from (3.11*a*) and (3.11*b*) is possible, nor the immediate transition between (3.2*a*), (3.2*b*) and (3.11*a*), (3.11*b*) is allowed, since the parameters Δ_0 and δ_0 are determined by the parities of the remaining parameters.

For $d_0 = \frac{1}{2}(L_1-\nu-\delta_0) > 0$, the summation parameters of A_β and $A_{\beta'}$ in (3.13) are restricted and the numerator–denominator polynomials $\bar{\mathbf{g}}_{k,\bar{\kappa}}$ include the common factors

$$\left(\tfrac{1}{2}(\nu + L_2 + \delta_0 + n - 2) - \overline{\kappa}\right)_{d_0+\overline{\kappa}-k} \left(\tau + \tfrac{1}{2}(\nu - L_1 + \delta_0) - \overline{\kappa} + 2\right)_{d_0+\overline{\kappa}-k}$$
$$\times[\tfrac{1}{2}(L_1 - \nu - \Delta_0 - 1) + \overline{\kappa}]^{(d_0+\overline{\kappa}-k)} \prod_{s=1}^{\overline{\kappa}-(\Delta_0+\delta_0)/2} \left(\tfrac{1}{2}(\nu + L_2 - \Delta_0 + n) - s\right)_{d_0}$$
$$\times \left(\tau + \tfrac{1}{2}(\nu - L_1 - \Delta_0) - s + 2\right)_{d_0} [\tfrac{1}{2}(L_1 - \nu + \delta_0 - 3) + s]^{(d_0)} \qquad (3.20a)$$

with the first three factors replaced by 1, when $d_0 + \overline{\kappa} - k < 0$. Although $\overline{\kappa}$ doesn't exceed $\sigma - d_0$, parameter k may be extended to the all values $k \leqslant \sigma$, in accordance with definitions (1.7*a*) or (1.7*b*). The numerator–denominator polynomials $\overline{\mathbf{g}}^{\overline{\kappa}k}$, respectively, include the common factors

$$\prod_{s=1}^{\overline{\kappa}-(\Delta_0+\delta_0)/2} [\tfrac{1}{2}(\nu + L_2 - \Delta_0 + n) - s]_{d_0+\theta_s-1}[\tau + \tfrac{1}{2}(\nu - L_1 - \Delta_0) - s + 2]_{d_0+\theta_s-1}$$
$$\times[\tfrac{1}{2}(L_1 - \nu + \delta_0 - 3) + s]^{(d_0+\theta_s-1)} \qquad (3.20b)$$

where $\theta_s = 1$ for $1 \leqslant s \leqslant k - (\Delta_0 + \delta_0)/2$ and 0 otherwise. The reduction factors (3.20*a*) and (3.20*b*) should be taken into account when rearranging the polynomial structure (distribution of the linear and polynomial factors under square root sign) of (3.11*a*) and (3.11*b*) for $\nu - L_1 + \delta_0 < 0$.

Note, that expressions (3.11*a*) and (3.11*b*) for the OCs are valid and may be proved separately for $\sigma = \frac{1}{2}(L_1 - L_2,),\ \tau = \frac{1}{2}\lambda$ and for $\sigma = \frac{1}{2}\lambda,\ \tau = \frac{1}{2}(L_1 - L_2)$, without using the substitutions (3.19). We see that polynomials (3.13) have the (truncated) SU(3) weight spaces of zeros [13, 17], correlated mutually and with linear factors in the numerator and denominator position of (3.11*a*), (3.11*b*) and coinciding with the roots of (3.20*a*) and (3.20*b*) (when condition (3.5) is relinquished). Besides, similarly as overlaps of the projected states, the polynomials $\overline{\mathbf{g}}_{k,\overline{\kappa}}$ and $\overline{\mathbf{g}}^{\overline{\kappa},k}$ are invariant with respect to the transformations, generated by the substitutions

$$L_1 \to -L_1 - n + 2 \qquad \text{or} \qquad L_2 \to -L_2 - n + 4 \qquad (3.21a)$$

and

$$L_1 \leftrightarrow L_2 - 1 \qquad (3.21b)$$

(together with $\delta_0 \leftrightarrow \delta_0$ for agreement of parities), but the total S_4 symmetry of the polynomials $\overline{\mathbf{g}}_{k,\overline{\kappa}}$ and $\overline{\mathbf{g}}^{\overline{\kappa},k}$ is correlated with the S_4 symmetry of four last double factorial factors in (3.12) under the square root sign. Thus, the uniqueness of expressions for the polynomials (3.13) are conditioned by the a', b', d', e' permutation symmetries, the distribution of zeros and the total powers of these polynomials, restricted by the number of squares in the partitions β, β' (cf [1, 2, 12, 16, 20]).

4. Concluding remarks

In this paper, the importance of the A_λ functions of the Biedenharn and Louck was demonstrated for the explicit solution of the orthogonalization problem of the biorthogonal projected bases of the different subgroup chains with the single missing label, such as SU(3) ⊗ SU(3) ⊃ SU(3) and SU(n) ⊃ SO(n). The corresponding approach, considered in section 2, may also be useful in the case of the complementary chain U(4) ⊃ U(2)⊕U(2), restricted to the block-diagonal subgroups [27], as well as for the chains U(n) $\supset$ U($n-2$)$\oplus$U(2) in the case of the three-parametric irreps. Probably, this approach may be also extended to explicit orthogonalization of the biorthogonal coupling coefficients [28] of quantum algebra $u_q(3)$, although its proof may be rather problematic. Otherwise, extension of some results of section 3 to the complementary chain Sp(4) $\supset$ U(2) (cf [2]) is also possible. Still an open problem is the possibility of the direct construction of the orthonormal states for these missing label states, in analogy with the explicit SU(3) $\otimes$ SU(3) $\supset$ SU(3) canonical solution [13, 20]. It is possible that the resolution of the problem of Louck and co-workers [10, 11] for SU(3) $\supset$ SO(3) may also appear as some contraction (with some hidden parameters, similar to Δ_0 and δ_0) of the SU(3) $\otimes$ SU(3) $\supset$ SU(3) canonical solution, with a related structure of the normalization factors and the denominator polynomials.

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Appendix. Orthogonalization coefficients for Elliott's basis of SU(3) $\supset$ SO(3)

Now we may specify our orthogonalization coefficients (3.11b) for the projected basis states (1.1a) of the SU(3) $\supset$ SO(3) Elliott basis, labelled by the K_+,

$$\delta + \Delta \leqslant K_+ \leqslant \min(\mu,\ L - \overline{\Delta}) \tag{A.1}$$

where $\delta = 0$ or 1, $\Delta = 0$ or 1, and $\overline{\Delta} = 0$ or 1, such that $\lambda + \mu - L - \delta,\ \lambda - L - \Delta$, and $\mu - L - \overline{\Delta}$ are even integers. Finally, for fixed μ we write

$$\mathcal{O}^{[\lambda\mu,L]_3}_{\widetilde{\kappa},K_+} = \frac{2\,(1+\delta_{K_+,0})^{-1}\,K_+^{(\Delta+\delta)/2}\,\widetilde{\mathbf{g}}^{\widetilde{\kappa},K_+}\,\mathbf{R}^{[\lambda\mu,L]_3}_{\widetilde{\kappa},K_+}}{\left[\frac{1}{2}(\widetilde{\kappa}-K_+)\right]!\left[\frac{1}{2}(\widetilde{\kappa}+K_+)\right]!\left[\widetilde{\mathbf{g}}^{\widetilde{\kappa},\widetilde{\kappa}}\,\widetilde{\mathbf{g}}^{\widetilde{\kappa}+2,\widetilde{\kappa}+2}\right]^{1/2}} \tag{A.2}$$

where on the r.h.s.

$$\mathbf{R}^{[\lambda\mu,L]_3}_{\widetilde{\kappa},K_+} = \left[\frac{(\lambda+\mu-L+\Delta-\widetilde{\kappa})!!(\lambda+\mu+L+\Delta-\widetilde{\kappa}+1)!!(\lambda+\mu+\delta-\widetilde{\kappa}+1)!}{\lambda!(\lambda+\mu+1)!(2L+1)}\right]^{1/2}$$
$$\times R\left(\tfrac{1}{2}K_+[L\ \overline{\Delta}]\tfrac{1}{2}\widetilde{\kappa};\ \tfrac{1}{2}\mu,\ \tfrac{1}{2}(L-\overline{\Delta})\right) \tag{A.3}$$

and the notation (3.4) with the concrete values $\sigma = \frac{1}{2}\mu$ and $\tau = \frac{1}{2}(L-\overline{\Delta})$ is used. The numerator–denominator polynomials (functions) may be expressed in (A.2) as follows:

$$\widetilde{\mathbf{g}}^{\widetilde{\kappa},K_+} = M(\beta'^*)\,A_{\beta'^*}\begin{pmatrix} -d'', -e'', -a'', -b'' \\ -c'' \end{pmatrix}$$
$$= (-1)^{\sum_s \beta'_s} M(\beta')\,A_{\beta'}\begin{pmatrix} a'', b'', d'', e'' \\ c'' \end{pmatrix} \tag{A.4}$$

with the partitions

$$\beta'^* = \left[\left(\tfrac{1}{2}(\widetilde{\kappa}-\delta-\Delta)\right)^{(\mu-\widetilde{\kappa})/2},\ \tfrac{1}{2}(K_+-\delta-\Delta)\right]$$
$$\beta' = \left[\left(\tfrac{1}{2}(\mu-\widetilde{\kappa})+1\right)^{(K_+-\delta-\Delta)/2},\ \left(\tfrac{1}{2}(\mu-\widetilde{\kappa})\right)^{(\widetilde{\kappa}-K_+)/2}\right] \tag{A.5}$$

Table A1. Polynomials $\widetilde{\mathbf{g}}^{\widetilde{\kappa},K_+}$ for fixed μ and multiplicities $\leqslant 4$.

$\widetilde{\kappa}$	K_+	$\widetilde{\mathbf{g}}^{\widetilde{\kappa},K_+}$	$\widetilde{\kappa}$	K_+	$\widetilde{\mathbf{g}}^{\widetilde{\kappa},K_+}$
Multiplicity = 2	$(\mu-\delta-\Delta=2)$				
μ	μ	$2^{-3}\omega_1(\lambda,L,\delta,\Delta)$	μ	$\delta+\Delta$	1 (for any multiplicity)
Multiplicity = 3	$(\mu-\delta-\Delta=4)$				
$\mu-2$	$\mu-2$	$2^{-6}\omega_2(\lambda,L,\delta,\Delta)$	$\mu-2$	$\mu-4$	$2^{-3}\omega_1(\lambda+1,L,\delta+1,\Delta+1)$
μ	μ	$2^{-6}\omega_2'(\lambda,L,\delta,\Delta)$	μ	$\mu-2$	$2^{-3}\omega_1(\lambda,L,\delta,\Delta)$
Multiplicity = 4	$(\mu-\delta-\Delta=6)$				
$\mu-4$	$\mu-4$	$2^{-9}\omega_3(\lambda,L,\delta,\Delta)$	$\mu-4$	$\mu-6$	$2^{-6}\omega_2(\lambda+1,L,\delta+1,\Delta+1)$
$\mu-2$	$\mu-2$	$2^{-12}H_4(\lambda,L,\delta,\Delta)$	$\mu-2$	$\mu-4$	$2^{-9}W_3(\lambda,L,\delta,\Delta)$
$\mu-2$	$\mu-6$	$2^{-6}\omega_2'(\lambda+1,L,\delta+1,\Delta+1)$	μ	μ	$2^{-9}\omega_3'(\lambda,L,\delta,\Delta)$
μ	$\mu-2$	$2^{-6}\omega_2'(\lambda,L,\delta,\Delta)$	μ	$\mu-4$	$2^{-3}\omega_1(\lambda,L,\delta,\Delta)$

Here

$$\omega_1(\lambda,L,\delta,\Delta)=(\lambda-\delta+2)^{(2)}(2\Delta+1)+(\lambda+\Delta-L+2)(\lambda+\Delta+L+3)(2\delta+1)$$

$$\begin{aligned}\omega_2(\lambda,L,\delta,\Delta)=&(\lambda-\delta+4)^{(4)}(2\Delta+3)^{(2;2)}\\&+2(\lambda-\delta+2)^{(2)}(2\Delta+1)(\lambda+\Delta-L+4)(\lambda+\Delta+L+5)(2\delta+1)\\&+(\lambda+\Delta-L+4)^{(2;2)}(\lambda+\Delta+L+5)^{(2;2)}(2\delta+3)^{(2;2)}\end{aligned}$$

$$\begin{aligned}\omega_2'(\lambda,L,\delta,\Delta)=&(\lambda-\delta+2)^{(4)}(2\Delta+3)^{(2;2)}\\&+2(\lambda-\delta+2)^{(2)}(2\Delta+3)(\lambda+\Delta-L+2)(\lambda+\Delta+L+3)(2\delta+3)\\&+(\lambda+\Delta-L+4)^{(2;2)}(\lambda+\Delta+L+5)^{(2;2)}(2\delta+3)^{(2;2)}\end{aligned}$$

$$\begin{aligned}\omega_3(\lambda,L,\delta,\Delta)=&(\lambda-\delta+6)^{(6)}(2\Delta+5)^{(3;2)}\\&+3(\lambda-\delta+4)^{(4)}(2\Delta+3)^{(2;2)}(\lambda+\Delta-L+6)(\lambda+\Delta+L+7)(2\delta+1)\\&+3(\lambda-\delta+2)^{(2)}(2\Delta+1)(\lambda+\Delta-L+6)^{(2;2)}(\lambda+\Delta+L+7)^{(2;2)}(2\delta+3)^{(2;2)}\\&+(\lambda+\Delta-L+6)^{(3;2)}(\lambda+\Delta+L+7)^{(3;2)}(2\delta+5)^{(3;2)}\end{aligned}$$

$$\begin{aligned}\omega_3'(\lambda,L,\delta,\Delta)=&(\lambda-\delta+2)^{(6)}(2\Delta+5)^{(3;2)}\\&+3(\lambda-\delta+2)^{(4)}(2\Delta+5)^{(2;2)}(\lambda+\Delta-L+2)(\lambda+\Delta+L+3)(2\delta+5)\\&+3(\lambda-\delta+2)^{(2)}(2\Delta+5)(\lambda+\Delta-L+4)^{(2;2)}(\lambda+\Delta+L+5)^{(2;2)}(2\delta+5)^{(2;2)}\\&+(\lambda+\Delta-L+6)^{(3;2)}(\lambda+\Delta+L+7)^{(3;2)}(2\delta+5)^{(3;2)}\end{aligned}$$

$$\begin{aligned}H_4(\lambda,L,\delta,\Delta)=&(\lambda-\delta+4)^{(6)}(\lambda-\delta+2)^{(2)}(2\Delta+5)^{(3;2)}(2\Delta+3)\\&+4(\lambda-\delta+4)^{(6)}(2\Delta+5)^{(3;2)}(\lambda+\Delta-L+4)(\lambda+\Delta+L+5)(2\delta+3)\\&+3(\lambda-\delta+4)^{(4)}(2\Delta+5)^{(2;2)}(\lambda+\Delta-L+4)^{(2;2)}(\lambda+\Delta+L+5)^{(2;2)}(2\delta+5)^{(2;2)}\\&+3(\lambda-\delta+2)^{(4)}(2\Delta+3)^{(2;2)}(\lambda+\Delta-L+6)^{(2;2)}(\lambda+\Delta+L+7)^{(2;2)}(2\delta+3)^{(2;2)}\\&+4(\lambda-\delta+2)^{(2)}(2\Delta+3)(\lambda+\Delta-L+6)^{(3;2)}(\lambda+\Delta+L+7)^{(3;2)}(2\delta+5)^{(3;2)}\\&+(\lambda+\Delta-L+6)^{(3;2)}(\lambda+\Delta-L+4)(\lambda+\Delta+L+7)^{(3;2)}(\lambda+\Delta+L+5)\\&\times(2\delta+5)^{(3;2)}(2\delta+3)\end{aligned}$$

$$\begin{aligned}W_3(\lambda,L,\delta,\Delta)=&(\lambda-\delta+4)^{(6)}(2\Delta+5)^{(3;2)}\\&+\tfrac{3}{2}(\lambda-\delta+4)^{(4)}(2\Delta+5)^{(2;4)}(\lambda+\Delta-L+4)(\lambda+\Delta+L+5)(2\delta+5)\\&+\tfrac{3}{2}(\lambda-\delta+2)^{(4)}(2\Delta+5)^{(2;4)}(\lambda+\Delta-L+4)(\lambda+\Delta+L+5)(2\delta+1)\\&+\tfrac{3}{2}(\lambda-\delta+2)^{(2)}(2\Delta+1)(\lambda+\Delta-L+6)^{(2;2)}(\lambda+\Delta+L+7)^{(2;2)}(2\delta+5)^{(2;4)}\\&+\tfrac{3}{2}(\lambda-\delta+2)^{(2)}(2\Delta+5)(\lambda+\Delta-L+4)^{(2;2)}(\lambda+\Delta+L+5)^{(2;2)}(2\delta+5)^{(2;4)}\\&+(\lambda+\Delta-L+6)^{(3;2)}(\lambda+\Delta+L+7)^{(3;2)}(2\delta+5)^{(3;2)}\end{aligned}$$

with $X^{(n;k)}=X(X-k)(X-2k)\cdots(X-nk+k)$ and $X^{(n)}=X^{(n;1)}$.

and parameters

$$\begin{aligned}&a''=\tfrac{1}{2}(\lambda+\overline{\Delta}-\delta+1)\qquad b''=\tfrac{1}{2}(\lambda-\overline{\Delta}-\delta+2)\\&c''=-\tfrac{1}{2}(2\lambda+\mu-\widetilde{\kappa})-2\qquad d''=\tfrac{1}{2}(\lambda+L-\Delta+2)\\&e''=\tfrac{1}{2}(\lambda-L-\Delta+1).\end{aligned}\tag{A.6}$$

In particular, $\widetilde{\mathbf{g}}^{\delta+\Delta,\delta+\Delta} = 1$, and for fixed $\mu - \delta - \Delta$, $\widetilde{\mathbf{g}}^{\mu,\delta+\Delta} = \widetilde{\mathbf{g}}^{\mu+2,\mu+2} = 1$. The concrete expressions of non-trivial polynomials $\widetilde{\mathbf{g}}^{\widetilde{\kappa},K_+}$ for multiplicities $\leqslant 4$ are presented in table 1. Invariance under the substitution $\overline{\Delta} \leftrightarrow 1-\overline{\Delta}$ allowed us to eliminate the explicit dependence of $\widetilde{\mathbf{g}}^{\widetilde{\kappa},K_+}$ on $\overline{\Delta}$.

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